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Gibbs States and Equilibrium States for Finitely Presented Dynamical Systems

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We study *finitely presented* dynamical systems (which generalize Axiom A systems) and show that the notions of equilibrium states and Gibbs states (for Hölder continuous functions) are equivalent. Our results extend those of Ruelle, Haydn, and others on Axiom A dynamical systems and statistical mechanics.

KEY WORDS: Dynamical systems; thermodynamic formalism; Gibbs states; invariant measures; symbolic dynamics; Markov partitions.

1. INTRODUCTION

The equivalence between equilibrium states and translation-invariant Gibbs states for lattice spin systems is a central theorem of statistical mechanics.[15] This result has been translated, with the help of symbolic dynamics, to the field of hyperbolic dynamical systems, more precisely, Axiom A diffeomorphisms.[4,12,15] In this article, we generalize this equivalence to *finitely presented systems*, a wider class of dynamical systems recently introduced by Fried.[11] This class contains in particular pseudo-Anosov homeomorphisms[10] and sofic shifts[5] and seems to be a natural generalization of Axiom A. Indeed, these systems possess very good symbolic dynamics and the methods from the thermodynamic formalism which apply to Axiom A diffeomorphisms carry over to this case. (For another generalization of Gibbs states see ref. 8.)

Precise definitions and results are stated in Section 2 (where we also prove a weak closing lemma and a weak spectral decomposition theorem for finitely presented systems). In Sections 3 and 4, we prove respectively the two directions of the equivalence (Theorems 1 and 2). In order to do

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this, we use the methods developed in previous papers[2,11,12,15] (for the sake of conciseness, we do not rewrite arguments which can be used *verbatim*, but give precise references). We recover (in Lemma 3.3) a known result[5] on sofic systems (a transitive sofic system with periodic points dense is intrinsically ergodic with support).

2. MEASURES FOR FINITELY PRESENTED SYSTEMS

We recall some of the theory of expansive homeomorphisms and Fried's[11] theory of finitely presented systems. Let Ω be a compact topological space and $f\colon \Omega \to \Omega$ an expansive homeomorphism, i.e., there is a closed neighborhood $V \subset \Omega \times \Omega$ of the diagonal Δ_Ω such that $F = f \times f\colon \Omega \times \Omega \to \Omega \times \Omega$ satisfies $\bigcap_{k \in \mathbb{Z}} F^{-k} V = \Delta_\Omega$.

This implies[11] (Lemma 2) that Ω is metrizable and that there exists a metric d for which f has an expansive constant $\varepsilon > 0$:

$$x, y \in \Omega, \qquad x \neq y \Rightarrow \exists k \in \mathbb{Z}, \qquad d(f^k x, f^k y) > \varepsilon \tag{2.1}$$

For sufficiently small $\delta > 0$, this metric can be chosen[11] such that it contracts (expands) the δ-stable (unstable) set uniformly, where those sets are defined for $x \in \Omega$ by

$$W_x^s(\delta) = \{ y \mid d(f^k x, f^k y) \leqslant \delta \ \forall k \geqslant 0 \}$$

$$W_x^u(\delta) = \{ y \mid d(f^k x, f^k y) \leqslant \delta \ \forall k \leqslant 0 \}$$

The contraction (expansion) property of the metric means that there exists $\lambda \in (0, 1)$ such that for all $x, y, z \in \Omega$, $y \in W_x^s(\delta)$, and $z \in W_x^u(\delta)$, one has

$$d(fz, fy) \leqslant \lambda d(x, y), \qquad d(f^{-1}x, f^{-1}z) \leqslant \lambda d(x, z) \tag{2.2}$$

We shall take $\delta < \varepsilon/2$. This implies that for any $x, y \in \Omega$, the intersection of $W_x^s(\delta)$ with $W_y^u(\delta)$ consists of at most one point. If we set

$$D_\delta = \{ (x, y) \in \Omega \times \Omega \mid W_x^s(\delta) \text{ meets } W_y^u(\delta) \}$$

and define $[\cdot, \cdot]\colon D_\delta \to \Omega$ by $[x, y] \in W_x^s(\delta) \cap W_y^u(\delta)$, then D_δ is closed in $\Omega \times \Omega$ and $[\cdot, \cdot]$ is continuous. We say that $R \subset \Omega$ is a rectangle if $R \times R \subset D_\delta$ and $[R, R] \subset R$. Then, for $x \in R$ we have the sets $W_\delta^s(x, R) = R \cap W_x^s(\delta)$ and $W_\delta^u(x, R) = R \cap W_x^u(\delta)$, and $R = [W_\delta^u(x, R), W_\delta^s(x, R)]$.

We now come to Fried's finitely presented dynamical systems. He has given four equivalent definitions of such systems. The expansive homeomorphism f satisfies the first one if Ω is a finite union of rectangles.

To state the second definition, we must recall the notion of a *Markov* *partition* for an expansive homeomorphism $f: \Omega \to \Omega$: it is a finite cover $\mathscr{R}$ of Ω by proper rectangles (R is proper if $R = \overline{\operatorname{int} R}$) with disjoint interiors, diameters less than δ, and such that, if $x \in \operatorname{int} R$, $fx \in \operatorname{int} R'$ (with $R, R' \in \mathscr{R}$), then

$$f(W^s_\delta(x, R)) \subset R' \qquad \text{and} \qquad f^{-1}(W^u_\delta(fx, R')) \subset R$$

If (for any $\delta > 0$) there exists a Markov partition for f, then f satisfies Fried's second definition.

For the third definition, we must introduce subshifts with finite symbol sets. Let $\mathscr{S}$ be a finite set and consider the shift map $\sigma(s_k) = (s_{k+1})$ on sequences $(s_k) \in \mathscr{S}^{\mathbb{Z}}$. The sequence space $\mathscr{S}^{\mathbb{Z}}$ endowed with the product topology is compact and metrizable and the shift is expansive. We say that a closed σ-invariant subset $\Sigma \subset \mathscr{S}^{\mathbb{Z}}$ is a *subshift of finite type*[9] (SFT) (of order two) defined by a *transition matrix* $M = (m_{ij})$, with $m_{ij} \in \{0, 1\}$ for all $i, j \in \mathscr{S}$, if $\Sigma = \{(s_k) \mid m_{s_k s_{k+1}} = 1, \forall k \in \mathbb{Z}\}$.

If the expansive homeomorphism $f: \Omega \to \Omega$ is a factor of a subshift of finite type Σ by a surjective semiconjugacy $\pi: \Sigma \to \Omega$, it then satisfies Fried's third definition.

We shall not need Fried's fourth definition and are going to use below whichever characterization is suitable, but mostly the existence of a Markov partition, of the corresponding subshift of finite-type extension and properties (2.1) and (2.2). [The relationship between the Markov partition and the subshift extension—i.e., the symbolic dynamics—is the same as in the Axiom A or Smale space setting,[15] Section 7.5. Observe that if we endow the subshift Σ with the metric $\tilde{d}((s_k), (t_k)) = \lambda^n$, where $n = \inf\{|k|, s_k \neq t_k\}$ and λ is the constant appearing in (2.2), then the projection π is Lipschitz.] In the sequel, $\mathscr{R} = \bigcup_{s \in \mathscr{S}} R(s)$ is a Markov partition for f of diameter at most δ and $\pi: \Sigma \to \Omega$ is the corresponding semiconjugacy between f and an SFT (ref. 11, p. 496).

Note that other authors have worked on expansive homeomorphisms with Markov partitions, in particular Dateyama[7] (see also the references therein).

We now define equilibrium states and Gibbs states for a continuous weight function $A: \Omega \to \mathbb{R}$. Here, we only use the fact that the space Ω is metric, compact, and the map $f: \Omega \to \Omega$ a homeomorphism; these definitions hence apply to the shift spaces.

An f-invariant Borel probability measure μ on Ω is an *equilibrium state*[20] for A (and the dynamical system f) if it realizes the following supremum:

$$h_f(\mu) + \int_\Omega A \, d\mu = \sup_\nu \left(h_f(\nu) + \int_\Omega A \, d\nu \right)$$

In the above equality, ν ranges in the set of all f-invariant Borel probability measures over Ω and $h_f(\nu)$ denotes the measure-theoretic entropy of f. It is known that the set of equilibrium states for an expansive homeomorphism is compact and nonempty (ref. 20, p. 224). Our results below show that if f is a topologically $+$-transitive, finitely presented system and A is Hölder continuous with exponent $\theta \in (0, 1)$ [note $A \in \mathscr{C}^\theta(\Omega)$], then there is a unique equilibrium state for A.

We now define the *abstract* notion of a Gibbs state.[4,15] We first need to know what a *conjugating map* is: for $\mathcal{O}$ a subset of Ω, $\varphi: \mathcal{O} \to \Omega$ is conjugating if

$$\lim_{|k| \to \infty} d(f^k(x), f^k(\varphi x)) = 0$$

uniformly for $x \in \mathcal{O}$. The sets $\mathcal{O}$ and $\varphi(\mathcal{O})$ will always be assumed to be compact subsets of Ω. If $\varphi: \mathcal{O} \to \varphi(\mathcal{O})$ is, moreover, a homeomorphism [for the induced metric on $\mathcal{O}$, $\varphi(\mathcal{O})$], one calls φ a conjugating homeomorphism.

We shall also say that $x, y \in \Omega$ are conjugate if $\lim_{|k| \to \infty} d(f^k(x), f^k(y)) = 0$.

Suppose now that A is an element of $\mathscr{C}^\theta(\Omega)$; we then say that a Borel probability measure μ on Ω is a *Gibbs state* for A [and (f, Ω)] if, for all conjugating homeomorphisms $\varphi: \mathcal{O} \to \varphi(\mathcal{O})$, one has $\varphi^*(g \cdot \mu|_{\mathcal{O}}) = \mu|_{\varphi(\mathcal{O})}$, where

$$g(x) = g_A(x) = \exp \sum_{k \in \mathbb{Z}} [A \circ f^k \circ \varphi(x) - A \circ f^k(x)]$$

We have slightly modified the usual[15] definition of Gibbs states by taking for $\mathcal{O}$ a compact subset instead of asking for an open subset (observe that in the subshift of finite-type setting, the natural sets for $\mathcal{O}$ are the open and closed cylinders).

In the Smale space or Axiom A setting, an equilibrium state for $A \in \mathscr{C}^\theta(\Omega)$ is a Gibbs state for A (ref. 15, Section 7.18). Conversely, Haydn has proved[12] that a Gibbs state for A is invariant under some iterate f^p of the map and is an equilibrium state for $\sum_{k=0}^{p-1} A \circ f^k$ (in particular, if f is topologically mixing—see ref. 15 for a definition—then $p = 1$). We obtain the same results:

Theorem 1. Let f be a finitely presented system on Ω and $A \in \mathscr{C}^\theta(\Omega)$. Every equilibrium state μ for A is a Gibbs state for A.

Theorem 2. Let f be a finitely presented system on Ω and $A \in \mathscr{C}^\theta(\Omega)$. For every Gibbs state μ for A, there exists an integer $p \in \mathbb{N}$

such that μ is invariant under f^p and is an equilibrium state for $A_p = \sum_{k=0}^{p-1} A \circ f^k$ and (f^p, Ω). In particular:

(1) If f is topologically $+$-transitive, every f-invariant Gibbs state μ for A is an equilibrium state for A.

(2) If f is topologically mixing, every Gibbs state μ for A is an equilibrium state for A.

The main difficulties encountered in finitely presented systems are due to the absence of a *spectral decomposition theorem* and the absence of a *shadowing lemma* (for instance, Walters[19] shows that the only subshifts which have the shadowing property are the subshifts of finite type; also, as noted in Dateyama,[7] pseudo-Anosov maps do not have the shadowing property). However, we have the following "weak closing lemma":

Lemma 2.1. Let f be finitely presented. The set $\mathrm{Rec}(f)$ of recurrent points for f is a subset of the closure $\overline{\mathrm{Per}(f)}$ of the set of periodic points.

Proof. By definition, $x \in \Omega$ is *recurrent* if $x \in \omega(x)$, where the ω-limit set of x, $\omega(x)$, is the set of all limit points of the sequence $f^k(x)$, $k \geqslant 0$. Let ε and λ be as in (2.1), (2.2), and let $0 < \beta < \varepsilon/2$. Let $x \in \mathrm{Rec}(f)$. We shall construct a sequence of periodic points converging to x. Let $0 < \delta < \beta \cdot (1 - \lambda)$ (this will be explained below). We shall need the constant $\eta = \eta(x, \delta)$ defined by

$$\eta = \sup\{\zeta \geqslant 0 \mid \{(x, y) \in \Omega \times \Omega \mid d(x, y) < \zeta\} \subset D_\delta\} \tag{2.3}$$

(Note that $\eta > 0$ because Ω is a finite union of rectangles.) Pick M so large that $\lambda^M \cdot \delta < \eta/2$ and choose $0 < \alpha < \eta/2$. Since x is recurrent, there exists $n \geqslant M$ such that $d(f^n x, x) < \alpha$. We construct an α-pseudo-orbit $(x_j)_{j=-\infty}^{\infty}$ by setting

$$x_j = f^k(x) \qquad \text{for} \quad j \equiv k (\mathrm{mod}\, n), \quad 0 \leqslant k < n$$

[Recall that a sequence (y_k) is an α-pseudo-orbit if $d(fy_k, y_{k+1}) < \alpha$ for all k.] For this very special pseudo-orbit, we now show that the usual shadowing construction applies (ref. 2, Proposition 3.6) to obtain a point which β-shadows it. In fact, it suffices to find $x^{(r \cdot n)}$ which β-shadows $(x_k)_{k=0}^{r \cdot n}$ for every $r \in \mathbb{N}$. (Because then $x^{\pm(r \cdot n)} := f^{-[(r \cdot n)/2]} x^{(r \cdot n)}$ will β-shadow $(x_k)_{k=-[(r \cdot n)/2]}^{[(r \cdot n)/2+1]}$ and the limit point z of the sequence $x^{\pm(r \cdot n)}$ for $r \to \infty$ clearly β-shadows $(x_k)_{k=-\infty}^{\infty}$.)

So let $r \geqslant 1$. We define x'_k for $k = 1, \dots, r \cdot n$ recursively by

$$\begin{cases} x'_0 = x_0 = x \\ x'_{k+1} \in W^u_\delta(f x'_k) \cap W^s_\delta(x_{k+1}) \end{cases}$$

We have to check that the intersection is not empty. For $0 \leqslant k < n-1$, there is no problem, since we clearly have $x'_{k+1} = x_{k+1} = f^{k+1}x$. Let us look at the first special case, $k+1 = n$:

$$x'_n \in W^u_\delta(fx'_{n-1}) \cap W^s_\delta(x_n) = W^u_\delta(f^n x_0) \cap W^s_\delta(x_0)$$

This point clearly exists because $d(f^n x_n, x_0) < \alpha < \eta/2$. If $r \geqslant 2$, we look at the next sequences of n points. Again, we have

$$x'_{k+1} = fx'_k, \quad \forall k \qquad \text{for which} \quad k+1 \not\equiv 0 \pmod{n}$$

[Because $fx'_k \in W^u_\delta(fx'_k) \cap W^s_\delta(x_{k+1})$, since $x'_k \in W^s_\delta(x_k)$ by induction and $x_{k+1} = f(x_k)$ for these values of k.]

For the other values of k, i.e., those for which $k+1 = l \cdot n$ $(l \geqslant 2)$, the intersection

$$x'_{l\cdot n} \in W^u_\delta(fx'_{l\cdot n-1}) \cap W^s_\delta(x_0)$$

is not empty either, because by induction

$$fx'_{l\cdot n-1} = f^n x'_{(l-1)\cdot n} \qquad \text{and} \qquad x'_{(l-1)\cdot n} \in W^s_\delta(x_0)$$

so that

$$d(f^n x'_{(l-1)\cdot n}, x_0) \leqslant d(x_0, f^n x_0) + d(f^n x_0, f^n x'_{(l-1)\cdot n}) < \alpha + \lambda^n \delta < \eta$$

We now claim that the point $x^{(r\cdot n)} := f^{-r\cdot n} x'_{r\cdot n}$ β-shadows $(x_k)_{k=0}^{r\cdot n}$. Indeed, if $0 \leqslant k \leqslant r \cdot n$,

$$\begin{aligned} d(f^k x^{(r\cdot n)}, x_k) &\leqslant d(f^k x^{(r\cdot n)}, x'_k) + d(x'_k, x_k) \\ &\leqslant \sum_{j=k+1}^{r\cdot n} d(f^{k-j} x'_j, f^{k-j+1} x'_{j-1}) + \delta \\ &< \delta \cdot \sum_{j=1}^{r\cdot n - k} \lambda^j + \delta < \frac{\delta}{1-\lambda} < \beta \end{aligned}$$

[Where we use $x'_j \in W^u_\delta(fx'_{j-1})$ and the choice of δ at the beginning of the proof.] But now, for z a limit point as described above and any $k \in \mathbb{Z}$,

$$\begin{aligned} d(f^k z, f^k f^n z) &\leqslant d(f^k z, x_k) + d(x_k, f^{k+n} z) \\ &= d(f^k z, x_k) + d(x_{k+n}, f^{k+n} z) \\ &< 2\beta < \varepsilon \end{aligned}$$

and hence $f^n z = z$ by expansivity. Since $d(x, z) < \beta$, the proof is finished. ∎

Remark. After writing this "weak shadowing" argument, we came across ref. 7, and realized that the fact we could shadow our special α-pseudo-orbit was a particular case of what is there called the "special pseudo-orbit tracing property" (SPOTP): indeed, Dateyama shows that SPOTP follows from the existence of a Markov partition. We shall also use the following result about nonwandering[15] points:

Lemma 2.2. If f is a finitely presented system, $B \subset \Omega$, and Λ_Σ is the set of nonwandering points of the Markov shift extension, then $B \subset \pi(\Lambda_\Sigma)$, if and only if $B \subset \overline{\mathrm{Per}(f)}$.

Proof. Assume first that $B \subset \pi(\Lambda_\Sigma)$. We have to show $B \subset \overline{\mathrm{Per}\, f}$. If $x \in B$, then by assumption $x = \pi(\tilde{x})$ with $\tilde{x} \in \Lambda_\Sigma$. For any neighborhood $\mathcal{O}$ of x, $\pi^{-1}\mathcal{O}$ is a neighborhood of $\tilde{x}$ and hence contains a periodic point $\tilde{p}$ [the equality $\Lambda_\Sigma = \overline{\mathrm{Per}(\sigma)}$ is well known in the SFT case and can be proved by using shadowing]. But then $\pi(\tilde{p}) \in \mathcal{O}$ is a periodic point.

Suppose now that $B \subset \overline{\mathrm{Per}\, f}$. Let $x \in B$. By hypothesis, $x = \lim p_i$, where p_i is a sequence of periodic points. But any $\tilde{p}_i \in \pi^{-1} p_i$ is a periodic point [this is due to the fact that $\#\pi^{-1}(x) \leqslant (\#\mathcal{S})^2$ and that is shown exactly as in the Smale space case—see e.g., Theorem IV.9.6.e in ref. 14]. So, taking if necessary a subsequence of $\tilde{p}_i$,

$$x = \lim p_i = \lim \pi(\tilde{p}_i) = \pi \lim(\tilde{p}_i)$$

so $\lim \tilde{p}_i \in \pi^{-1}x$ is the limit of a sequence of periodic points, hence nonwandering. ∎

Applying Lemma 2.2 to $\mathrm{Rec}(f)$ and using compactness of $\pi(\Lambda_\Sigma)$, we obtain:

Corollary 2.3. Let f be a finitely presented system. We have the inclusions:

(1) $\mathrm{Per}(f) \subset \mathrm{Rec}(f) \subset \overline{\mathrm{Per}(f)}$ and hence $\overline{\mathrm{Rec}(f)} = \overline{\mathrm{Per}(f)}$.

(2) $\mathrm{Rec}\, f \subset \pi(\Lambda_\Sigma)$ and hence $\overline{\mathrm{Rec}(f)} \subset \pi(\Lambda_\Sigma)$.

In particular, if Λ denotes the nonwandering set of f, we have

$$\overline{\mathrm{Rec}(f)} = \overline{\mathrm{Per}(f)} \subset \pi(\Lambda_\Sigma) \subset \Lambda$$

However, we have neither been able to deduce from the definition of a finitely presented system the equality $\Lambda = \pi(\Lambda_\Sigma)$, or, equivalently, the "strong closing lemma" $\Lambda = \overline{\mathrm{Per}(f)}$, nor found an example where they do not hold. These equalities are always satisfied in Smale spaces (Anosov closing lemma) and for pseudo-Anosov homeomorphisms (because they are topologically mixing—since the associated Markov shift is Bernoulli—and because of Lemma 2.4 below). This point should be clarified.[2]

We end this section with a weak spectral decomposition theorem. We first introduce the set $K=\bigcup_{k\in\mathbb{Z}} f^k(\partial R)$. [Note that π^{-1} is uniquely defined on the residual set $\Omega\backslash K$; see ref. 11, p. 496. One also easily proves $\Lambda\backslash\pi(\Lambda_\Sigma)\subset K$.]

Lemma 2.4. Let f be a finitely presented system. There is a finite compact cover of $\overline{\mathrm{Rec}(f)}$ by f-invariant sets, $\overline{\mathrm{Rec}(f)}\subset\bigcup_{i=1}^{r}\Lambda_i\subset\Lambda$, such that $\Lambda_i\cap\Lambda_j\subset K$ if $i\neq j$, and $f|_{\Lambda_i}$ is topologically $+$-transitive. Moreover, each Λ_i has a finite compact cover $\Lambda_i=\bigcup_{j=1}^{d_i}\Lambda_{i,j}$, where the $\Lambda_{i,j}$ may only meet in K, are f^{d_i} invariant, cyclically permuted by f, and such that $f^{d_i}|_{\Lambda_{i,j}}$ is topologically mixing.

Proof. The spectral decomposition theorem for subshifts of finite type (ref. 9, Proposition 17.11) yields disjoint topologically $+$-transitive subshifts of finite type $\tilde{\Lambda}_1,\ldots,\tilde{\Lambda}_r$ such that $\Lambda_\Sigma=\bigcup\tilde{\Lambda}_i$. The $\tilde{\Lambda}_i$ are called basic sets. Moreover, each $\tilde{\Lambda}_i$ can be decomposed in $d_i\geqslant 1$ disjoint compact sets $\tilde{\Lambda}_{i,j}$ which are cyclically permuted by σ and such that $\sigma^{d_i}|_{\tilde{\Lambda}_{i,j}}$ is topologically mixing.

Clearly, f leaves the compact set $\Lambda_i=\pi(\tilde{\Lambda}_i)$ invariant and $f|_{\Lambda_i}$ is topologically $+$-transitive (for an example of nonempty intersections, see ref. 5, Remark 4, Section 5). Moreover, Corollary 2.3 tells us that $\overline{\mathrm{Rec}(f)}\subset\pi(\Lambda_\Sigma)$. Finally, the compact sets $\pi(\tilde{\Lambda}_{i,j})$ are f^{d_i} invariant and f^{d_i} is mixing on them (for the nonempty intersections, see ref. 5, Remark 3, Section 5). ∎

3. EQUILIBRIUM STATES ARE GIBBS STATES

In the Smale space case, Theorem 1 is the "easy" direction of the equivalence. We are going to use the available symbolic dynamics to follow the method outlined in ref. 15. Since our hypotheses are weaker, we first check that some key results remain true: the fact that the boundary of the Markov partition can be divided into a stable and an unstable boundary and the isomorphism between the abstract dynamical systems $(\Sigma,\sigma,\tilde{\mu})$ and (Ω,f,μ), where $\tilde{\mu}$ is the unique equilibrium state for a given $A\in\mathscr{C}^\theta(\Omega)$ and $\mu=\pi^*(\tilde{\mu})$. This is the content of the next two lemmas and their corollaries.

[2] This situation can be compared with the Axiom A case: Dankner[(6)] has constructed an example of a diffeomorphism g in $\mathbb{R}^3$ with hyperbolic nonwandering set Λ in which the periodic points are *not* dense, i.e., g satisfies the first but not the second condition of Axiom A. We do not know if the expansive homeomorphism $g|_\Lambda$ is finitely presented. Kurata[(13)] has a simpler example with the same properties on a four-dimensional manifold. Another idea would be to try to construct a sofic shift counterexample.

Lemma 3.1. Let $R \subset \Omega$ be any closed rectangle. As a subset of Ω, R has boundary $\partial R = \partial^u R \cup \partial^s R$, where

$$\partial^u R = \{x \in R \mid x \notin \operatorname{int} W^s_\delta(x, R)\}$$
$$\partial^s R = \{x \in R \mid x \notin \operatorname{int} W^u_\delta(x, R)\}$$

and the interiors are taken in $W^s_x(\delta)$, respectively $W^u_x(\delta)$.

Proof. (This lemma should be compared with Lemma 3.11 of ref. 2, although we are not allowed to use the product $[\cdot, \cdot]$ as freely.) One inclusion is easy: if x lies in the interior of R, then $x \in \operatorname{int} W^u_\delta(x, R)$ and $x \in \operatorname{int} W^s_\delta(x, R)$. For the other direction, suppose $x \in \partial R$ and use Fried's observation (ref. 11, p. 497) that this implies that $x \in R$ and there exists a sequence $x_k \to x$ $(k \to \infty)$ with $x_k \notin R$ and such that either $x_k \in W^u_x(\delta)$, but then $x \notin \operatorname{int} W^u_\delta(x, R)$; or $x_k \in W^s_x(\delta)$, and then $x \notin \operatorname{int} W^s_\delta(x, R)$. ∎

We use the notation $\partial^s \mathcal{R} = \bigcup_{t \in \mathcal{S}} \partial^s R(t)$, $\partial^u \mathcal{R} = \bigcup_{t \in \mathcal{S}} \partial^u R(t)$. Lemma 3.1 yields:

Corollary 3.2. $f(\partial^s \mathcal{R}) \subset \partial^s \mathcal{R}$ and $f^{-1} \partial^u \mathcal{R} \subset \partial^u \mathcal{R}$.

Proof. Use Lemma 3.1 and follow Bowen's proof (ref. 2, Lemma 3.14 and Proposition 3.15), which only employs the existence of canonical coordinates inside rectangles. ∎

The next lemma is essentially the analogue of Ruelle's Theorem 7.9.[15]

Lemma 3.3. Assume that f is topologically $+$-transitive.

(1) σ is topologically $+$-transitive. If, moreover, f is topologically mixing, then σ is topologically mixing.

(2) If $A \in \mathscr{C}(\Omega)$, then $P(f, A) = P(\sigma, A \circ \pi)$.

(3) If $A \in \mathscr{C}^\theta(\Omega)$, then there exists a unique equilibrium state μ for A and $\mu = \pi^*(\tilde{\mu})$, where $\tilde{\mu}$ is the unique equilibrium state for $A \circ \pi$.

(4) Let μ, $\tilde{\mu}$ be the measures defined in (3). The map $\pi: (\Sigma, \tilde{\mu}) \to (\Omega, \mu)$ is an isomorphism of dynamical systems. [In particular, $\tilde{\mu}(\pi^{-1} K) = 0$.]

(5) The set of periodic points for f is dense in Ω.

(6) If f is topologically mixing, the set of points conjugate to any $x \in \Omega$ is dense in Ω.

Proof. For (1) the proof of Bowen (ref. 2, Proposition 3.19) is valid. For the assertions (2)–(4), remember that if we put the right metrics $\tilde{d}$ on Σ and d on Ω, then π is Lipschitz [in particular, $A \in \mathscr{C}^\theta(\Omega)$ implies $A \circ \pi \in$

$\mathscr{C}^\theta(\Sigma)$]. We can hence use Corollary 3.2 to proceed exactly as in ref. 15 (Sections 7.7–7.9). For (5), use (1) and Lemma 2.4. The last claim is true because it holds in the shift space and one can use (1) (see ref. 15, 7.16.b). ▮

Remark. Lemma 3.3(1) combined with Lemma 2.2 shows that the equality $\overline{\mathrm{Per}(f)} = \Lambda$ is true in particular if f is topologically +-transitive.

A consequence of the lemma, which permits us to suppose $A = 0$, is:

Corollary 3.4. Assume f is topologically +-transitive. If $A, B \in \mathscr{C}^\theta(\Omega)$, denote by μ_A and μ_{A+B} the equilibrium states of Lemma 3.3 for A, respectively $A + B$, and write

$$Z_{n,m} = \mu_A\left(\exp \sum_{k=n}^{m-1} B \circ f^k\right), \qquad n < m \in \mathbb{Z}$$

One has

$$\lim_{\substack{n \to -\infty \\ m \to \infty}} \frac{1}{Z_{n,m}} \cdot \left(\exp \sum_{k=n}^{m-1} B \circ f^k\right) \cdot \mu_A = \mu_{A+B}$$

where the convergence is meant in the sense of the vague topology.

Proof. Lemma 2.4 allows us to restrict ourselves to the topologically mixing case. We can hence follow ref. 15 (Corollary 7.13.b). ▮

To prove Theorem 1, we must understand conjugating homeomorphisms better. In the Smale space (or SFT) setting, the situation is described in refs. 4 and 15. Unfortunately, an apparently crucial point [condition (C) in ref. 16: for every pair of conjugate points there is a conjugating homeomorphism defined on an open neighborhood of the first and sending the first to the second] does not seem true in our setting (because we are not allowed to use $[\cdot, \cdot]$ freely). But some results apply directly:

Lemma 3.5.

(1) Two points $x, y \in \Omega$ are conjugate if and only if there is $N \in \mathbb{N}$ such that $d(f^k x, f^k y) \leqslant \varepsilon$ for all $|k| > N$.

(2) A mapping $\varphi: \mathcal{O} \to \Omega$ is conjugating if and only if there is $N \in \mathbb{N}$ such that $d(f^k x, f^k(\varphi x)) \leqslant \varepsilon$ for all $|k| > N$ and $x \in \mathcal{O}$.

(3) Let $x \in \mathcal{O} \subset \Omega$ and $\varphi: \mathcal{O} \to \Omega$ be a conjugating map, continuous at x. Then there is a neighborhood $\mathcal{O}'$ of x such that $\varphi|_{\mathcal{O}' \cap \mathcal{O}}$ is injective and continuous.

(4) If two mappings φ_1 and φ_2 defined on $\mathcal{O}$ are conjugating, con-

tinuous at $x \in \mathcal{O}$, and such that $\varphi_1(x) = \varphi_2(x)$, then there is a neighborhood $\mathcal{O}'$ of x such that $\varphi_1|_{\mathcal{O}' \cap \mathcal{O}} = \varphi_2|_{\mathcal{O}' \cap \mathcal{O}}$.

Proof. The proofs are in ref. 4: points 3.3–3.6. Note that if the sets $\mathcal{O}$ of (3), (4) are neighborhoods of x, then $\mathcal{O} \cap \mathcal{O}'$ is also a neighborhood of x. ∎

Lemma 3.5 allows us to speak of germs of conjugating homeomorphisms at a point. As already mentioned, the difference with refs. 4, 12, and 15 is that such a germ is not associated to *any* pair of conjugate points. However, we are going to prove that the set of "bad" points is negligible in the measure-theoretic sense. In order to do this, we introduce yet another notation. For $x \in \Omega \backslash K$ we define two symbolic sequences in $\mathcal{S}^{\mathbb{Z}}$:

(1) The first one is simply $s_k(x) = (\pi^{-1}x)_k$ [i.e., $f^k(x) \in \operatorname{int} R(s_k)$ for $k \in \mathbb{Z}$].

(2) For the second, choose for each $k \in \mathbb{Z}$ one of the rectangles R of the Markov partition $[R \neq R(s_k)]$ which minimize the (strictly positive) distance $d(f^k x, R)$, and set $s'_k(x) = s$, where $R = R(s)$.

The "bad" set (which is essentially the set Y of Theorem 1 in ref. 17) is

$$\begin{aligned} Y := \{x \in \Omega \mid \lim_{k \to \infty} d(f^k x, \partial\mathcal{R}) = 0 \text{ or } \lim_{k \to -\infty} d(f^k x, \partial\mathcal{R}) = 0\} \\ = K \cup \{x \in \Omega \backslash K \mid \lim_{k \to \infty} d(f^k x, R(s'_k(x))) = 0 \\ \text{or } \lim_{k \to -\infty} d(f^k x, R(s'_k(x))) = 0\} \end{aligned}$$

Y is a closed subset of Ω, it contains K, and its complement is

$$\begin{aligned} \Omega \backslash Y = \{x \in \Omega \backslash K \mid \exists \eta = \eta(x) > 0 \text{ such that } \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, n \geqslant N, \\ \text{with } d(f^n x, R(s'_n(x))) > \eta \text{ and } d(f^{-n} x, R(s'_{-n}(x))) > \eta\} \end{aligned} \tag{3.1}$$

We now prove:

Lemma 3.6. The complement of $Y \backslash \partial\mathcal{R}$ is of total measure, i.e., if μ is an f-invariant Borel probability measure, then $\mu(Y \backslash \partial\mathcal{R}) = 0$.

Proof. We can write $Y \backslash \partial\mathcal{R}$ as $Y^+ \cup Y^-$, where

$$Y^+ = \{x \in \Omega \backslash \partial\mathcal{R} \mid \lim_{k \to \infty} d(f^k x, \partial\mathcal{R}) = 0\}$$

$$Y^- = \{x \in \Omega \backslash \partial\mathcal{R} \mid \lim_{k \to -\infty} d(f^k x, \partial\mathcal{R}) = 0\}$$

It clearly suffices to study Y^+. First note that Y^+ is a subset of the set of nonrecurrent points $\{x \in \Omega \mid x \notin \omega(x)\}$. This comes from the observation that, for $y \in Y^+$, $\omega(y) \subset \partial\mathscr{R}$, but $y \notin \partial\mathscr{R}$. The proof now follows from Poincaré's recurrence theorem (ref. 14, Theorem I.2.3), which tells us that $\mu(\{x \mid x \notin \omega(x)\}) = 0$, if μ is a Borel f-invariant probability measure. ∎

Remarks. 1. If μ is an equilibrium state for some $A \in \mathscr{C}^\theta(\Omega)$, we know $\mu(\partial\mathscr{R}) \leqslant \mu(K) = 0$ and Lemma 3.6 implies $\mu(Y) = 0$.

2. If $x \in Y$ and $y \in \Omega$ are conjugate, then y is also in Y. [Consider the case $y \in Y^+ \cup \partial\mathscr{R}$. This means that $\lim_{k \to \infty} d(f^k x, \partial\mathscr{R}) = 0$. Since by assumption $\lim_{k \to \infty} d(f^k x, f^k y) = 0$, we have $\lim_{k \to \infty} d(f^k y, \partial\mathscr{R}) = 0$, i.e., $y \in Y^+ \cup \partial\mathscr{R}$.]

In the sequel, we shall use the following equivalence relations: two Markov rectangles $R(s)$ and $R(t)$ are *related* if $R(s) \cap R(t) \neq \emptyset$; two sequences (s_k) and (t_k) in Σ are *related* if $R(s_k)$ and $R(t_k)$ are related for all $k \in \mathbb{Z}$. Bowen has observed (ref. 3, p. 13) that $\pi(s_k) = \pi(t_k)$ if and only if (s_k) and (t_k) are related.

We now construct conjugating homeomorphisms for conjugate points outside Y:

Lemma 3.7. Suppose $x \in \Omega \setminus Y$, $y \in \Omega$ are conjugate.

(1) There exists a compact neighborhood $\mathcal{O}$ of x and a conjugating homeomorphism $\varphi\colon \mathcal{O} \to \varphi(\mathcal{O})$ such that $\varphi(x) = y$.

(2) $\pi^{-1}x$ and $\pi^{-1}y$ are conjugate in Σ.

Proof. (1) Let $\eta(x)$ be as in (3.1). Since x and y are conjugate, there exists N such that $d(f^k x, f^k y) < \eta(x)/2$ for all $|k| \geqslant N$. Then, since x is in $\Omega \setminus Y$, there exists $n \in \mathbb{N}$, $n \geqslant N$ such that

$$\begin{cases} f^n(x) \in \operatorname{int} R(s_n) & \text{and } d(f^n(x), \partial R(s_n)) > \eta(x) \\ f^{-n}(x) \in \operatorname{int} R(s_{-n}) & \text{and } d(f^{-n}(x), \partial R(s_{-n})) > \eta(x) \end{cases}$$

which implies $f^n(y) \in \operatorname{int} R(s_n)$ and $f^{-n}(y) \in \operatorname{int} R(s_{-n})$.

Thus, since $y \notin \partial\mathscr{R}$ by Remark 2 above, we can now use Ruelle's construction (ref. 15, Section 7.15), which we repeat here for the reader's convenience.

Denote by R_y the rectangle of the Markov partition such that $y \in \operatorname{int} R_y$. Since the Markov partition is finite, there exists a neighborhood $\mathcal{O}$ of x such that $[x, z]$ and $[z, x]$ are well defined for z in $\mathcal{O}$. By choosing $\mathcal{O}$ sufficiently small, we can assume that $f^n[z, x]$ and $f^{-n}[x, z]$ lie in $R(s_n)$, respectively $R(s_{-n})$, for $z \in \mathcal{O}$. Hence, we can construct

$$z_1 = [f^n[z, x], f^n y], \qquad z_2 = [f^{-n} y, f^{-n}[x, z]]$$

and, replacing if necessary $\mathcal{O}$ by a smaller neighborhood, assume that z_1, z_2 are in a neighborhood of $f^n y$, respectively $f^{-n}y$, small enough to ensure that $f^{-n}z_1 \in R_y$ and $f^n z_2 \in R_y$. Hence, for $z \in \mathcal{O}$, we can set

$$\varphi(z) := [f^{-n}z_1, f^n z_2] = [f^{-n}[f^n[z, x], f^n y], f^n[f^{-n}y, f^{-n}[x, z]]]$$

Clearly, φ is continuous at x and is a conjugating map in $\mathcal{O}$. We now use Capocaccia's methods to show that a suitable restriction of φ is a homeomorphism: by Lemma 3.5(3), we can suppose that φ is injective and continuous; since $y \notin Y$, we can construct an injective continuous conjugating map $\varphi' : \mathcal{O}' \to \varphi'\mathcal{O}'$ sending y to x ($\mathcal{O}'$ is a suitable neighborhood of y). But now $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are continuous and conjugating maps fixing y, respectively x, and defined in neighborhoods of these points: uniqueness [Lemma 3.5(4)] allows us to conclude that they are equal to the identity map on suitable neighborhoods of x, y.

(2) Since x and y are conjugate, denoting $(t_k) = \pi^{-1}(y)$ and $(s_k) = \pi^{-1}(x)$, the rectangles $R(s_k)$ and $R(t_k)$ are related for $|k| \geqslant N$ [N as in (1)] and also $s_n = t_n$, $s_{-n} = t_{-n}$ for some $n \geqslant N$. Hence,

$$\begin{aligned} x &= \pi(..., s_{-n-1}, s_{-n}, s_{-n+1}, ..., s_0, ..., s_n, s_{n+1}, ...) \\ &= \pi(..., t_{-n-1}, t_{-n}, s_{-n+1}, ..., s_0, ..., t_n, t_{n+1}, ...) \end{aligned}$$

But since $x \notin K$, we must have $s_k = t_k$, for all $|k| \geqslant n$, and this exactly means that $\pi^{-1}x$ and $\pi^{-1}y$ are conjugate. ∎

Now that we understand better the finitely presented situation, we are going to follow ref. 15, Theorem 7.17 in the next two lemmas to prove Theorem 1. First, we show that the equilibrium state for $A = 0$ (i.e., the measure which maximizes entropy) has a local product structure:

Lemma 3.8. Suppose f is topologically $+$-transitive and let μ be the unique probability measure which maximizes entropy. Then, for each $x \in \Omega$, the set

$$[W_x^u(\delta), W_x^s(\delta)] = \{[y, z] \mid (y, z) \in (W_x^u(\delta) \times W_x^s(\delta)) \cap D_\delta\}$$

is a neighborhood of x.

In this neighborhood, there is a local product structure for μ, i.e., there are positive measures μ_x^s on $W_x^s(\delta)$ and μ_x^u on $W_x^u(\delta)$ such that

$$[\cdot, \cdot]^* \, (\mu_x^u \times \mu_x^s)|_{(W_x^u(\delta) \times W_x^s(\delta)) \cap D_\delta} = \mu|_{[W_x^u(\delta), W_x^s(\delta)]}$$

Proof. The first observation comes from the fact that there is a neighborhood of x composed of finitely many rectangles, $R_1, ..., R_n$, say.

Decomposing $W^u_x(\delta) = \bigcup_j W^u_x(\delta, R_j)$ and $W^s_x(\delta) = \bigcup_j W^s_x(\delta, R_j)$ and remembering that $[W^u_x(\delta, R_j), W^s_x(\delta, R_j)] = R_j$, we obtain the assertion.

For the rest, we restrict ourselves for simplicity to the unstable set case and use Sinai's ideas[18] as presented in ref. 17. Rereading the proof of Theorem 1 in ref. 17, we notice that the only point which is delicate in the absence of canonical coordinates is the use of the "projection along unstable manifolds" defined in that paper. Fortunately, it suffices for our purpose to consider these projections

$$p^u_{y'}\colon \quad W^u_y(\alpha) \to W^u_{y'}(\delta)$$

when $y, y' \in W^u_x(\delta)$, $d(y, y') < \alpha$, $\alpha > 0$ sufficiently small (because we only use the projections to show that "the definition of μ^u_x makes sense" and not to compare μ^u_x and $\mu^u_{x'}$ for any x close to x'—in particular, we do *not* claim Ruelle and Sullivan's Theorem 1b is true in our setting). However, in this case $p^u_{y'}$ is simply the identity map and Lemma 3.8 follows from the proof in ref. 17. (For more details, see the Appendix in ref. 1.) ∎

The local product structure allows us to prove Theorem 1 in the case $A = 0$ (this is the method used in ref. 15, 7.17.b):

Lemma 3.9. Suppose f is topologically $+$-transitive and let μ be the unique probability measure which maximizes entropy. Then, if $\varphi\colon \mathcal{O} \to \varphi(\mathcal{O})$ is a conjugating homeomorphism, the image by φ of μ restricted to $\mathcal{O}$ is μ restricted to $\varphi(\mathcal{O})$:

$$\varphi^*(\mu|_{\mathcal{O}}) = \mu|_{\varphi\mathcal{O}}$$

Proof. One can clearly assume that $\mathcal{O}$ is contained in a small neighborhood of x. Let y denote $\varphi(x)$ and apply Lemma 3.8 to x.

It clearly suffices to show that, taking if necessary a smaller value of $\delta > 0$,

$$\begin{cases} \varphi^*(\mu^u_x|_{\mathcal{O} \cap \varphi^{-1}(W^u_y(\delta))}) = \mu^u_y|_{\varphi(\mathcal{O} \cap W^u_x(\delta))} \\ \varphi^*(\mu^s_x|_{\mathcal{O} \cap \varphi^{-1}(W^s_y(\delta))}) = \mu^s_y|_{\varphi(\mathcal{O} \cap W^s_x(\delta))} \end{cases} \tag{3.2}$$

The details of the proof can be found in the Appendix of ref. 1. ∎

Proof of Theorem 1. If μ is an equilibrium state for A, μ is f-invariant, and by Poincaré's recurrence theorem, its support is a subset of $\mathrm{Rec}(f) \subset \pi(\Lambda_\Sigma)$. (Use Corollary 2.3.) We then use the decomposition in basic sets Λ_i for f given by Lemma 2.4. Note that if $x \in \Lambda_i$ and $x \notin Y$, any point y conjugate to x is in the same basic set Λ_i. [Because, by Lemma 3.7(2), $\pi^{-1}(x) \in \tilde{\Lambda}_i$ and $\pi^{-1}y$ are conjugate and the stated result is true for SFT, i.e., $\pi^{-1}y$ must lie in $\tilde{\Lambda}_i$. See also ref. 15, 7.16.b.]

So, by Lemma 3.6, it suffices to prove the theorem for a topologically $+$-transitive $f: \Omega \to \Omega$: Lemma 3.9 tells us that the statement holds for $A = 0$ and Corollary 3.4 shows that it is true for any $A \in \mathscr{C}^\theta(\Omega)$. ∎

4. GIBBS STATES ARE EQUILIBRIUM STATES

To prove Theorem 2, we are going to follow some of Haydn's[12] ideas. As in the preceding section, let us start with some necessary results from the Axiom A case which obviously remain true for finitely presented systems.

Lemma 4.1.

(1) For $\delta > 0$, let M_δ be the greatest integer such that any y, y' in Ω with $d(y, y') < \delta$ satisfy $d(f^k y, f^k y') < \varepsilon$, $\forall\, |k| \leqslant M_\delta$. Then $M_\delta \to \infty$ as $\delta \to 0$.

(2) For $M \in \mathbb{N}$, let δ_M be the smallest number such that any y, y' in Ω with $d(f^k y, f^k y') < \varepsilon$, $\forall\, |k| \leqslant M$, satisfy $d(y, y') < \delta_M$. Then $\delta_M \to 0$ as $M \to \infty$.

(3) Let $\tilde{\mathcal{O}} \subset \Sigma$ be a compact set and $\tilde{\varphi}: \tilde{\mathcal{O}} \to \tilde{\varphi}(\tilde{\mathcal{O}})$ be a uniformly continuous conjugating homeomorphism for σ. Then there exist a finite decomposition $\tilde{\mathcal{O}} = \bigcup_k \tilde{\mathcal{O}}_k$ and conjugating homeomorphisms $\varphi_k: \mathcal{O}_k := \pi(\tilde{\mathcal{O}}_k) \to \varphi_k(\mathcal{O}_k)$ which are projections of $\tilde{\varphi}\,|_{\tilde{\mathcal{O}}_k}$ (i.e., $\varphi_k \circ \pi = \pi \circ \tilde{\varphi}$ on $\tilde{\mathcal{O}}_k$, for all k). The sets $\tilde{\mathcal{O}}_k$ can be assumed to be cylinders and the sets $\mathcal{O}_k$ are hence closed.

(4) If $\tilde{\mu}$ is a Gibbs state for $\tilde{A} \in \mathscr{C}^\theta(\Sigma)$, then $\tilde{\mu}(\Lambda_\Sigma) = 1$, where Λ_Σ denotes the nonwandering set for σ.

Proof. The first two assertions are Lemma 3 in ref. 12 or points 3.1 and 3.2 in ref. 4. The first one comes from the fact that f^k is uniformly continuous for all $k \in \mathbb{Z}$, and the second is due to expansiveness of f and compactness of Ω. The third claim is Lemma 4 in ref. 12 and, with the help of (1) and (2), this proof is valid in our case. The last assertion is Haydn's Lemma 9.[12] ∎

We now want to follow Haydn's steps and prove that $\mu(K) = 0$ if μ is a Gibbs state.

Lemma 4.2. Let $x \in \Omega$ be such that $\pi^{-1} x \subset \Lambda_\Sigma$. Then there exist: a finite closed cover $\bigcup_{l=1}^{n_x} \mathcal{O}_l$ of a neighborhood $\mathcal{O}$ of x such that $x \in \mathcal{O}_l$ for each l; and conjugating homeomorphisms $\varphi_l: \mathcal{O}_l \to \varphi_l(\mathcal{O}_l)$ with $\varphi_l(x) \notin K$ for each l.

Proof. Let $\{\tilde{x}_l\}_{l=1}^{n_x} := \pi^{-1}(x) \subset \Lambda_\Sigma$. Fix $\tilde{x}_l$ in this set and let $\tilde{\Lambda}^l_{i,j}$ be the mixing compact set of the spectral decomposition of σ which contains $\tilde{x}_l$ (see Lemma 2.4). Let $C_{\tilde{x}_l}$ be the set of points in Σ conjugate to $\tilde{x}_l$. Since $\sigma^{d_i}|_{\tilde{\Lambda}^l_{i,j}}$ is topologically mixing, $C_{\tilde{x}_l}$ is dense in $\tilde{\Lambda}^l_{i,j}$ (ref. 15, Proposition 7.16.b). Now the set $B_{\tilde{x}_l} := \pi^{-1}(\Omega \backslash K) \cap \tilde{\Lambda}^l_{i,j}$ is open in $\tilde{\Lambda}^l_{i,j}$ and nonempty [we cannot have $\pi(\tilde{\Lambda}^l_{i,j}) \subset K$ because of Lemma 3.3(4) and the fact that $\tilde{\mu}(\tilde{\Lambda}^l_{i,j}) \neq 0$ for any equilibrium state $\tilde{\mu}$ on Σ]. Hence, $C_{\tilde{x}_l} \cap B_{\tilde{x}_l}$ is not empty: choose a point $\tilde{y}_l$ in this intersection. [In particular, $\pi(\tilde{y}_l) \notin K$.] By construction, $\tilde{x}_l$ and $\tilde{y}_l$ are conjugate; we can hence find a compact neighborhood $\tilde{\mathcal{O}}_l$ of $\tilde{x}_l$ and a (uniformly continuous) conjugating homeomorphism $\tilde{\varphi}_l: \tilde{\mathcal{O}}_l \to \tilde{\varphi}_l(\tilde{\mathcal{O}}_l)$ with $\tilde{\varphi}_l(\tilde{x}_l) = \tilde{y}_l$ (ref. 15, Section 7.15).

We now apply Lemma 4.1(3) to $\tilde{\varphi}_l$, taking $\mathcal{O}_l = \mathcal{O}_{l,k}$, $\varphi_l := \varphi_{l,k}$ with k chosen such that $x \in \mathcal{O}_{l,k}$. We finish by noting that the set $\mathcal{O} = \bigcup \mathcal{O}_l$ is a neighborhood of x because it is of the type $\mathcal{O} = \pi(\bigcup_{\tilde{x}_l \in \pi^{-1}(x)} \tilde{C}_l)$, where each $\tilde{C}_l$ is a cylinder containing $\tilde{x}_l$. Indeed, if N is the maximal length of these cylinders, $\mathcal{O}$ contains the set

$$U := \bigcup_{\substack{(\prod_{j=-N}^{N-1} m_{s_j s_{j+1}}) = 1 \\ f^j x \in R(s_j)}} R(s_0) \cap f^{-1} R(s_1) \cap f R(s_{-1}) \cap \cdots \cap f^{-N} R(s_N) \cap f^N R(s_{-N})$$

which is clearly a neighborhood of x. [Let $\zeta > 0$ be such that any $y \in \Omega$ with $d(x, y) < \zeta$ satisfies $d(f^j x, f^j y) < \inf^+ \{\operatorname{diam} \mathscr{R}, d(f^j x, \partial \mathscr{R})\}$, for all $|j| \leqslant N$, where $\inf^+ B := \inf(B \backslash \{0\})$. Then $B_\zeta(x)$, the open ball of radius ζ and center x, is contained in U, since $d(x, y) < \zeta$ implies that $f^j y$ is in the same rectangle as $f^j x$ for $|j| \leqslant N$.] ∎

The next lemma is an adaptation of ref. 12, Lemma 5:

Lemma 4.3. Let $A \in \mathscr{C}^\theta(\Omega)$ and μ be an f-invariant Gibbs state on Ω. Then $\mu(K) = 0$.

Proof. We begin like Haydn and note that the proof can be reduced to showing $\mu(K^*) = 0$, where $K^* := \bigcap_{k \geqslant 0} f^k \partial^s \mathscr{R}$ (the set $\bigcap_{k \leqslant 0} f^k \partial^u \mathscr{R}$ would be treated similarly). In fact, using Poincaré's recurrence theorem and Lemma 2.4, it suffices to consider the set $K^* \cap \bigcup \Lambda_i$ [where $\bigcup \Lambda_i \subset \overline{\operatorname{Rec}(f)}$ is the union of the basic sets]. In particular, considering the finitely presented system $f|_{\bigcup \Lambda_i}$, we may assume by Lemma 3.3(1) that $\pi^{-1} \cup \Lambda_i \subset \Lambda_\Sigma$. Assume by contradiction that $\mu(K^* \cap \overline{\operatorname{Rec}(f)})$ is strictly positive and use Haydn's observation that this implies, since $K^* \cap \overline{\operatorname{Rec}(f)}$ is compact, the existence of $z \in K^* \cap \overline{\operatorname{Rec}(f)}$ such that $\mu(B_\zeta(z) \cap K^*) > 0$ for all $\zeta > 0$.

We now apply Lemma 4.2, which gives us a finite closed cover $\bigcup \mathcal{O}_l$ of $B_\zeta(z)$ for some $\zeta > 0$ and conjugating homeomorphisms $\varphi_l: \mathcal{O}_l \to \varphi_l(\mathcal{O}_l)$

sending z to some $w_l \notin K$. Since $w_l \notin K^*$, we can find $\eta > 0$ such that $B_{2\eta}(w_l) \cap K^* = \emptyset$ for all l and, taking if necessary a smaller value of ζ, $\varphi_l(B_\zeta(z) \cap \mathcal{O}_l) \subset B_\eta(w_l)$. Fix l such that $\pi(B_\zeta(z) \cap \mathcal{O}_l \cap K^*) > 0$ (this clearly exists).

We can now finish the proof like Haydn: set $D = \varphi_l(B_\zeta(z) \cap K^* \cap \mathcal{O}_l)$. Then $\mu(D) > 0$ because μ is a Gibbs state. Since φ_l is conjugating, there exists $N \in \mathbb{N}$ such that $d(f^k \varphi_l(y), f^k y) < \eta$, for all $y \in B_\zeta(z) \cap \mathcal{O}_l$ and all $|k| \geqslant N$. Hence, since K^* is f-invariant, $\sup_{x \in D} d(f^k(x), K^*) < \eta$ for $|k| \geqslant N$. But by construction, $d(D, K^*) > \eta$ and thus $f^k(D) \cap D = \emptyset$ for all $|k| \geqslant N$.

We have constructed a collection $\{f^{jN}(D), j \in \mathbb{N}\}$ of pairwise disjoint sets which all have the same strictly positive measure (because μ is f-invariant). But then the measure of their union diverges, a contradiction. ∎

Corollary 4.4. Let $A \in \mathscr{C}^\theta(\Omega)$ and μ be a Gibbs state on Ω for A. Then $\mu(K) = 0$.

Proof. Use Lemma 4.3 and follow Haydn (ref. 12, Proposition 6). ∎

Proof of Theorem 2. Let μ be a Gibbs state for $A \in \mathscr{C}^\theta(\Omega)$. By Corollary 4.4, π^{-1} is defined μ-almost everywhere. Define a measure $\tilde{\mu}$ on Σ by $\tilde{\mu}(U) = 0$ if $U \subset \pi^{-1}(K)$ and $\tilde{\mu}(U) = \tilde{\mu}(U \cap \pi^{-1}(\Omega \backslash K)) = \mu(\pi(U))$ for all other U. Using Lemma 4.1(3), one sees that $\tilde{\mu}$ is a Gibbs state for $A \circ \pi \in \mathscr{C}^\theta(\Sigma)$.

We first consider the special cases (1) and (2): we know that an f-invariant Gibbs state for a topologically $+$-transitive SFT or an *a priori* not invariant Gibbs state for a mixing SFT is an equilibrium state (ref. 15, Corollary 5.6 and Proposition 5.20, or ref. 12, Corollary 13), so $\tilde{\mu}$ is an equilibrium state for $A \circ \pi$. We end the proof of the special cases by applying Lemma 3.3, which says that $\mu = \pi^* \tilde{\mu}$ is an equilibrium state for A.

For the general case, apply Lemma 4.1(4) and then use the decomposition into mixing sets in the shift space (Lemma 2.4). Clearly, a normalization of $\tilde{\mu}|_{\tilde{\Lambda}_{i,j}}$ is invariant under σ^{d_i} and is an equilibrium state for $\sum_{k=0}^{d_i - 1} A \circ \pi \circ \sigma^k$ and $(\sigma^{d_i}, \tilde{\Lambda}_{i,j})$ (see, e.g., the proofs of Lemma 12 and Theorem 2 in ref. 12). Hence, we obtain the assertion of the theorem by setting p to be the least common multiple of the d_i, and applying again Lemma 3.3. ∎

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